

Analytic Minkowski Functionals of the Cosmic Microwave Background: Second-order Non-Gaussianity with Bispectrum and Trispectrum

Takahiko Matsubara*

Department of Physics, Nagoya University, Chikusa, Nagoya, 464-8602, Japan

(Dated: March 4, 2010)

Analytic formulas of Minkowski functionals in two-dimensional random fields are derived, including effects of second-order non-Gaussianity in the presence of both the bispectrum and trispectrum. The set of formulas provides a promising method to constrain the primordial non-Gaussianity of the universe by temperature fluctuations in the cosmic microwave background radiation. In a case of local-type non-Gaussianity, the Minkowski functionals are analytically given by powers of quadratic and cubic parameters, f_{NL} and g_{NL} . Our formulas are not restricted to this particular model, and applicable to a wide class of non-Gaussian models. The analytic formulas are compared to numerical evaluations from non-Gaussian realizations of temperature maps, showing very good agreements.

PACS numbers: 98.80.-k, 98.70.Vc, 98.80.Jk, 98.80.Es

I. INTRODUCTION

Detecting the primordial non-Gaussianity will play a key role in discriminating the models of the early universe. All models of inflation predict the primordial non-Gaussianity to some extent in principle. While a simple inflationary model with a single slow-rolling scalar field produces too small non-Gaussianity to be observed [1–6], many other models such as the curvaton scenario [7–9], models with non-standard kinetic term [10, 11], cyclic or ekpyrotic universes without inflation [13], etc. predict large non-Gaussianity which could be detected in any future [14].

The statistical properties of random Gaussian fields are completely characterized by the two-point correlation function in configuration space, or the power spectrum in Fourier space. Thus the information of non-Gaussianity is contained in higher-order polyspectra, such as the bispectrum, trispectrum, and so forth. Recently, much work is devoted to analyzing the temperature fluctuations in the cosmic microwave background (CMB) to constrain the primordial non-Gaussianity [15–20]. So far most of the constraints are placed to a phenomenological parameter f_{NL} , which is largely responsible for the primordial bispectrum. However, the bispectrum is not sufficient to fully discriminate the wide variety of models of the early universe. Some models, such as the curvaton scenario [21–23], ekpyrotic universes [24], etc. can generate a large trispectrum with a relatively small bispectrum. The trispectrum is often characterized by phenomenological parameters g_{NL} (and τ_{NL} in some models). When g_{NL} is large enough such that $g_{\text{NL}} \gg f_{\text{NL}}^2$, the bispectrum is not significant and the primordial non-Gaussianity can be detected only through the effect of trispectrum.

Straightforward measurements of the polyspectra become progressively complicated for higher-order statistics beyond the bispectrum, since the higher-order polyspectra have many arguments with complicated dependence on shapes and scales. Even though direct calculations of the higher-order

polyspectra may be primary methods to constrain higher-order non-Gaussianity, it is desirable to have as many alternative methods as possible. Among others, the set of Minkowski functionals (MFs) [25–27] has been proved to be a simple and powerful tool in analyzing non-Gaussianity in CMB [15, 16, 18, 28–33].

Since the hierarchy of polyspectra contains all the information of the statistical property of random field, the expectation values of MFs should be expressible by the polyspectra. Such relations were analytically derived for weakly non-Gaussian fields in the lowest-order approximation [34]. It was shown that deviations of the non-Gaussian MFs from the Gaussian MFs are proportional to a linear combination of the bispectrum. The analytic formulas are incorporated with the CMB bispectrum [35] and are applied to the analysis of CMB, successfully constraining non-Gaussianity parameter f_{NL} without relying on massive numerical simulations to generate non-Gaussian CMB maps [36–38].

So far the analytic formulas of non-Gaussian MFs are known only in the lowest-order approximation, which we call first-order non-Gaussianity. The contributions from the trispectrum and other polyspectra to the geometrical descriptors, such as the MFs, appear in higher-order non-Gaussianities [34, 39, 40]. To extract any information from MFs beyond the bispectrum, such as g_{NL} and τ_{NL} , it is particularly useful to have analytic formulas with higher-order approximations. The purpose of this paper is to derive the analytic formulas with second-order approximation of non-Gaussianity, including the effects up to the trispectrum. These primary formulas are applicable to a wide class of non-Gaussian fields in 2-dimensional space. We also focus on how the formulas are applied to analyses of CMB temperature fluctuations. Assuming a local model of non-Gaussianity, we derive the explicit dependence of the MFs on parameters f_{NL} and g_{NL} (and τ_{NL}).

This paper is organized as follows: In §2, the analytic formulas of MFs with second-order non-Gaussianity are derived. Non-Gaussianity parameters to describe the analytic MFs are explicitly given by the bispectrum and trispectrum in §3. In §4, the polyspectra in the local model of non-Gaussianity are summarized for applications to the analytic MFs. Numerical

*Electronic address: taka@phys.nagoya-u.ac.jp

tests of the analytic formulas are given in §5, and our results are summarized in §6.

II. ANALYTIC MINKOWSKI FUNCTIONALS WITH SECOND-ORDER NON-GAUSSIANITY

The MFs are morphological descriptors to study statistical properties of random fields. For a smooth scalar field $f(\mathbf{x})$ with zero mean, $\langle f \rangle = 0$, the excursion set over a given threshold ν , $Q = \{\mathbf{x} | f(\mathbf{x}) > \nu\sigma\}$ are defined, where $\sigma \equiv \langle f^2 \rangle^{1/2}$ is the rms of the field. The MFs for a set Q in a 2-dimensional space with smooth boundary ∂Q are given by [27]

$$V_0(\nu) = \int_Q da, \quad V_1(\nu) = \frac{1}{4} \int_{\partial Q} d\ell, \quad V_2(\nu) = \frac{1}{2\pi} \int_{\partial Q} \kappa d\ell, \quad (1)$$

where da and $d\ell$ denote the surface element of Q and the line element along ∂Q , respectively, and κ is the geodesic curvature on the boundary ∂Q . Apart from numerical factors, Minkowski functionals V_0, V_1, V_2 correspond to the area of Q , the length of ∂Q , and the curvature integral of ∂Q , respectively. According to the Gauss-Bonnet theorem, 2π times the Euler characteristic χ of Q is equal to the integral of geodesic curvature of ∂Q plus the integral of Gaussian curvature of Q . Therefore, when the excursion set is defined on the sphere of radius R , the Euler characteristic is given by a linear combination of the Minkowski functionals via $\chi = V_2 + V_0/2\pi R^2$ [27]. In a flat space ($R \rightarrow \infty$), $\chi = V_2$.

The expectation values of MFs in 2D are generically given by analytic forms [34]

$$V_k(\nu) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_2}{\omega_{2-k}\omega_k} \left(\frac{\sigma_1}{\sqrt{2}\sigma} \right)^k e^{-\nu^2/2} v_k(\nu), \quad (2)$$

where $\sigma_1 \equiv \langle (\nabla u)^2 \rangle^{1/2}$ is the variance of the gradient field, and $\omega_k \equiv \pi^{k/2}/\Gamma(k/2 + 1)$ is the volume of the unit ball in k dimensions, i.e., $\omega_0 = 1, \omega_1 = 2, \omega_2 = \pi$. For a random Gaussian field, the MFs are analytically given by Tomita's formula [41], which corresponds to

$$v_k(\nu) = H_{k-1}(\nu), \quad (3)$$

where

$$H_n(\nu) = e^{\nu^2/2} \left(-\frac{d}{d\nu} \right)^n e^{-\nu^2/2}, \quad (4)$$

are the Hermite polynomials, and we define a function

$$H_{-1}(\nu) \equiv e^{\nu^2/2} \int_\nu^\infty dv e^{-v^2/2} = \sqrt{\frac{\pi}{2}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}} \right), \quad (5)$$

for $k = 0$ in Eq. (3).

Throughout this paper, we assume hierarchical orderings of the higher-order correlators, $\langle f^n \rangle_c \sim \mathcal{O}(\sigma^{2n-2})$, where $\langle \cdots \rangle_c$ denotes the connected part, or the cumulant. In such a case, the reduced MFs $v_k(\nu)$ are expanded by the variance σ as

$$v_k = v_k^{(0)} + v_k^{(1)}\sigma + v_k^{(2)}\sigma^2 + \cdots \quad (6)$$

where $v_k^{(0)} = H_{k-1}(\nu)$ is the Gaussian contribution to the MFs. Primordial non-Gaussianities generated by most of the inflationary models satisfy the hierarchical orderings which we assume. In the previous work [34], the first-order corrections to the MFs in a non-Gaussian field are analytically derived:

$$v_k^{(1)}(\nu) = \frac{S}{6} H_{k+2}(\nu) - \frac{kS_I}{4} H_k(\nu) - \frac{k(k-1)S_{II}}{4} H_{k-2}(\nu), \quad (7)$$

where

$$S = \frac{\langle f^3 \rangle}{\sigma^4}, \quad S_I = \frac{\langle f^2 \nabla^2 f \rangle}{\sigma^2 \sigma_1^2}, \quad S_{II} = \frac{2\langle |\nabla f|^2 \nabla^2 f \rangle}{\sigma_1^4}, \quad (8)$$

are the skewness parameter and its derivatives.

Fundamental techniques to evaluate the next-order term, $v_k^{(2)}$, are mostly found in [34]. We outline below the higher-order extensions of the calculation in [34]. First we define the normalized cumulants of the field derivatives,

$$M_{\mu_1 \cdots \mu_n}^{(n)} \equiv \sigma^{2-2n} \langle f_{\mu_1} \cdots f_{\mu_n} \rangle_c \quad (9)$$

(which corresponds to the quantity denoted by $\hat{M}_{\mu_1 \cdots \mu_n}^{(n)}$ in [34]), where f_μ denotes all the field derivatives, $(f_\mu) = (f, f_{;1}, f_{;2}, f_{;11}, f_{;22}, f_{;12}, \dots)$, and “;” indicates the covariant derivative. In the following we adopt a notation, $f_0 = f$, $f_1 = f_{;1}$, $f_2 = f_{;2}$, $f_{11} = f_{;11}$, and the corresponding index runs over $\mu = 0, 1, 2, 11$. The Eq. (9) is of zero-th order in σ for hierarchical orderings. For any statistic which is locally defined by some function F of field derivatives f_μ , the expectation value of the statistic is expanded as (Eq. (22) of [34])

$$\begin{aligned} \langle F \rangle &= F^G + \frac{1}{6} \sum M_{\mu_1 \mu_2 \mu_3}^{(3)} F_{\mu_1 \mu_2 \mu_3}^G \sigma \\ &+ \left[\frac{1}{24} \sum M_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4)} F_{\mu_1 \mu_2 \mu_3 \mu_4}^G \right. \\ &\quad \left. + \frac{1}{72} \sum M_{\mu_1 \mu_2 \mu_3}^{(3)} M_{\mu_4 \mu_5 \mu_6}^{(3)} F_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6}^G \right] \sigma^2 \\ &+ \mathcal{O}(\sigma^3), \end{aligned} \quad (10)$$

where $F^G \equiv \langle F \rangle_G$, $F_{\mu_1 \mu_2 \cdots}^G \equiv \langle \partial_{\mu_1} \partial_{\mu_2} \cdots F \rangle_G$, $\partial_\mu = \partial/\partial f_\mu$, and $\langle \cdots \rangle_G$ denotes the expectation value for the Gaussian field which has the same two-point functions as the non-Gaussian field we consider (see [34] for detail). The first two terms on RHS of Eq. (10) correspond to the Gaussian contribution and first-order non-Gaussian corrections, respectively.

For the MFs, the function F is given by [34]

$$F = \begin{cases} \Theta(u - \nu), & (k = 0), \\ \frac{\pi}{8} \delta(u - \nu) |u_1|, & (k = 1), \\ -\frac{1}{2} \delta(u - \nu) \delta(u_1) |u_2| |u_{11}|, & (k = 2), \end{cases} \quad (11)$$

where Θ, δ are the step function and the delta function, respectively, and $u_\mu = f_\mu/\sigma$. The quantities $F_{\mu_1 \mu_2 \cdots}^G$ are explicitly

calculated to give [34]

$$F_{\mu_1\mu_2\cdots}^G = \frac{e^{-v^2/2}}{(2\pi)^{(k+1)/2}} q^{k-l_1-l_2-2m} \times \begin{cases} H_{n-1}(v)\delta_{l_1 0}\delta_{l_2 0}\delta_{m 0}, & (k=0), \\ \frac{\pi}{4}H_n(v)h_{l_1-2}\delta_{l_2 0}\delta_{m 0}, & (k=1), \\ h_{l_1}h_{l_2-2}[H_{n+1}(v)\delta_{m 0} - H_n(v)\delta_{m 1}] & (k=2), \end{cases} \quad (12)$$

where $q \equiv \sigma_1/\sqrt{2}\sigma$, and n, l_1, l_2, m are numbers of 0, 1, 2, 11, respectively, in a set of indices μ_1, μ_2, \dots . The factor h_l is defined by

$$h_l = \begin{cases} 0, & (l : \text{odd}), \\ (-2)^{l/2}\pi^{-1/2}\Gamma[(l+1)/2], & (l : \text{even}). \end{cases} \quad (13)$$

For example, $h_{-2} = 1$, $h_{-1} = 0$, $h_0 = 1$, $h_1 = 0$, $h_2 = -1$, $h_3 = 0$, $h_4 = 3$, etc.

The factor $M_{\mu_1\mu_2\mu_3}^{(3)}$ is expressible by the skewness and its derivatives of Eq. (8) from rotational symmetry [34]. The non-zero components are

$$M_{000}^{(3)} = S, \quad M_{00(11)}^{(3)} = q^2 S_I, \\ M_{011}^{(3)} = M_{022}^{(3)} = -\frac{q^2}{2} S_I, \quad M_{22(11)}^{(3)} = q^4 S_{II} \quad (14)$$

and their permutations. For other combinations of the indices, $M_{\mu_1\mu_2\mu_3}^{(3)} = 0$. Substituting Eq. (12) and Eq. (14) into the second term in the RHS of Eq. (10), the first-order corrections to the MFs of Eq. (7) follow.

It is straightforward to extend the above calculation to derive second-order corrections. We only need to calculate 4-point cumulants $M_{\mu_1\mu_2\mu_3\mu_4}^{(4)}$. With similar considerations of rotational symmetry in the case of 3-point cumulants, we obtain

$$M_{0000}^{(4)} = K, \quad M_{000(11)}^{(4)} = q^2 K_I, \\ M_{0011}^{(4)} = M_{0022}^{(4)} = -\frac{q^2}{3} K_I, \quad M_{011(11)}^{(4)} = -q^4 K_{III}, \\ M_{022(11)}^{(4)} = q^4 (K_{II} - K_{III}), \quad M_{1111}^{(4)} = M_{2222}^{(4)} = 3q^4 K_{III}, \\ M_{1122}^{(4)} = q^4 K_{III}. \quad (15)$$

For other combinations of the indices which cannot be obtained by permutations in the above equations, $M_{\mu_1\mu_2\mu_3\mu_4}^{(4)} = 0$. Kurtosis parameter and its derivatives are defined by

$$K = \frac{\langle f^4 \rangle_c}{\sigma^6}, \quad K_I = \frac{\langle f^3 \nabla^2 f \rangle_c}{\sigma^4 \sigma_1^2}, \\ K_{II} = \frac{2\langle f |\nabla f|^2 \nabla^2 f \rangle_c + \langle |\nabla f|^4 \rangle_c}{\sigma^2 \sigma_1^4}, \quad K_{III} = \frac{\langle |\nabla f|^4 \rangle_c}{2\sigma^2 \sigma_1^4}. \quad (16)$$

Substituting Eq. (12)–(15) into the third term in the RHS of

Eq. (10), we obtain the second-order corrections to the MFs:

$$v_0^{(2)}(v) = \frac{S^2}{72} H_5(v) + \frac{K}{24} H_3(v), \quad (17)$$

$$v_1^{(2)}(v) = \frac{S^2}{72} H_6(v) + \frac{K - S S_I}{24} H_4(v) \\ - \frac{1}{12} \left(K_I + \frac{3}{8} S_I^2 \right) H_2(v) - \frac{K_{III}}{8}, \quad (18)$$

$$v_2^{(2)}(v) = \frac{S^2}{72} H_7(v) + \frac{K - 2S S_I}{24} H_5(v) \\ - \frac{1}{6} \left(K_I + \frac{1}{2} S S_{II} \right) H_3(v) - \frac{1}{2} \left(K_{II} + \frac{1}{2} S_I S_{II} \right) H_1(v). \quad (19)$$

These equations are our primary results of this paper. In Eqs. (2), (6), (7), (17)–(19), the MFs of non-Gaussian fields are analytically given by the skewness, kurtosis, and their derivatives in second-order approximations. The above formulas are applicable to any 2D random field which has hierarchical orderings of the higher-order cumulants.

III. RELATION TO THE BISPECTRUM AND TRISPECTRUM

Once the skewness, kurtosis and their derivatives are calculated from a cosmological model, the prediction of MFs follows from our formulas. The skewness and its derivatives are directly given by the bispectrum, and the kurtosis and its derivatives are directly given by the trispectrum.

For a field on a 2D flat space, such as the flat-sky approximation of the CMB temperature fluctuations $\Delta T(\theta)/T$, one can adopt the Fourier transform:

$$\frac{\Delta T}{T}(\theta) = \int \frac{d^2 l}{(2\pi)^2} a(l) e^{i l \cdot \theta}. \quad (20)$$

The Fourier coefficients of the smoothed field f is given by $\tilde{f}(l) = a(l)W(l)$, where $W(l)$ is the window function of the smoothing kernel, where we assume spherically symmetric smoothing window. For a Gaussian window with smoothing angle θ_s , $W(l) = e^{-l^2 \theta_s^2/2}$. The power spectrum C , the bispectrum B , and the trispectrum T of the 2D field $\Delta T/T$ are defined by

$$\langle a(l_1) a(l_2) \rangle_c = (2\pi)^2 \delta^2(l_1 + l_2) C(l_1), \quad (21)$$

$$\langle a(l_1) a(l_2) a(l_3) \rangle_c = (2\pi)^2 \delta^2(l_1 + l_2 + l_3) B(l_1, l_2, l_3), \quad (22)$$

$$\langle a(l_1) a(l_2) a(l_3) a(l_4) \rangle_c = (2\pi)^2 \delta^2(l_1 + l_2 + l_3 + l_4) \\ \times T(l_1, l_2, l_3, l_4; l_{12}, l_{23}), \quad (23)$$

where $l_i = |l_i|$, $l_{ij} = |l_i + l_j|$. The translational symmetry is guaranteed by delta functions, and the rotational symmetry is taken into account in above definitions of arguments. By straightforward calculations using symmetric permutations in

integrals, one can show that

$$\sigma_j^2 = \int \frac{l_1 dl_1}{2\pi} l_1^j C(l_1) W^2(l_1), \quad (24)$$

$$S_A = \frac{1}{\sigma^4} \int \frac{l_1 dl_1}{2\pi} \frac{l_2 dl_2}{2\pi} \frac{d\theta_{12}}{2\pi} \tilde{S}_A(l_1, l_2) \times B(l_1, l_2, l_{12}) W(l_1) W(l_2) W(l_{12}), \quad (25)$$

$$K_A = \frac{1}{\sigma^6} \int \frac{l_1 dl_1}{2\pi} \frac{l_2 dl_2}{2\pi} \frac{l_3 dl_3}{2\pi} \frac{d\theta_{12}}{2\pi} \frac{d\theta_{23}}{2\pi} \tilde{K}_A(l_1, l_2, l_3) \times T(l_1, l_2, l_3, l_4; l_{12}, l_{23}) W(l_1) W(l_2) W(l_3) W(l_4), \quad (26)$$

where

$$l_{12} = (l_1^2 + l_2^2 + 2l_1 l_2 \cos \theta_{12})^{1/2}, \quad (27)$$

$$l_{23} = (l_2^2 + l_3^2 + 2l_2 l_3 \cos \theta_{23})^{1/2}, \quad (28)$$

$$l_4 = [l_{12}^2 + l_{23}^2 - l_2^2 + 2l_1 l_3 \cos(\theta_{12} + \theta_{23})]^{1/2}, \quad (29)$$

and $\sigma_0 = \sigma$, $(S_A) = (S, S_I, S_{II})$, $(\tilde{S}_A) = (\tilde{S}, \tilde{S}_I, \tilde{S}_{II})$, $(K_A) = (K, K_I, K_{II}, K_{III})$, $(\tilde{K}_A) = (\tilde{K}, \tilde{K}_I, \tilde{K}_{II}, \tilde{K}_{III})$,

$$\tilde{S} = 1, \quad \tilde{S}_I = -\frac{l_1^2}{2q^2}, \quad \tilde{S}_{II} = \frac{l_1^2(l_1^2 - 2l_2^2)}{4q^4}, \quad (30)$$

$$\tilde{K} = 1, \quad \tilde{K}_I = -\frac{l_1^2}{2q^2}, \quad \tilde{K}_{II} = \frac{l_{12}^4 - 4l_1^2 l_3^2}{16q^4}, \quad (31)$$

$$\tilde{K}_{III} = \frac{l_{12}^4 + 4l_1^2(l_3^2 - l_{12}^2)}{32q^4}. \quad (32)$$

The integration ranges are $2 \leq l_i \leq \infty$ and $0 \leq \theta_{ij} < 2\pi$ when the monopole and dipole components are subtracted from the map [see Eqs. (53)-(55) and explanations below].

For a field on a 2D sphere, such as the all-sky CMB map $\Delta T(\theta, \phi)/T$, one can adopt the expansion by spherical harmonics:

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{lm} a_{lm} Y_l^m(\theta, \phi). \quad (33)$$

The harmonic coefficients of the smoothed field f is given by $f_{lm} = a_{lm} W_l$, where W_l is the window function. For a Gaussian window with smoothing angle θ_s , $W_l = e^{-l(l+1)\theta_s^2/2}$. When monopole and dipole components are subtracted as usually in the case of CMB map, $W_0 = W_1 = 0$. The power spectrum, bispectrum and trispectrum are defined by [42–44]

$$\langle a_{l_1 m_1} a_{l_2 m_2} \rangle_c = (-1)^{m_1} \delta_{l_1 l_2} \delta_{m_1, -m_2} C_{l_1}, \quad (34)$$

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle_c = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{l_1 l_2 l_3}, \quad (35)$$

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4} \rangle_c = \sum_{L, M} (-1)^M \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ m_3 & m_4 & M \end{pmatrix} T_{l_3 l_4}^{l_1 l_2}(L), \quad (36)$$

where $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3j-symbol, and the rotational symmetry is taken into account in the above definitions. The

symmetries of the trispectrum $T_{l_3 l_4}^{l_1 l_2}(L)$ are enforced by the reduced trispectrum $\mathcal{T}_{l_3 l_4}^{l_1 l_2}(L)$ which is an arbitrary function of its arguments except that it must be symmetric against exchange of its upper and lower indices $\mathcal{T}_{l_3 l_4}^{l_1 l_2}(L) = \mathcal{T}_{l_1 l_2}^{l_3 l_4}(L)$ [44]. The construction is

$$T_{l_3 l_4}^{l_1 l_2}(L) = P_{l_3 l_4}^{l_1 l_2}(L) + (2L+1) \sum_{L'} \left[(-1)^{l_2+l_3} \begin{Bmatrix} l_1 & l_2 & L \\ l_4 & l_3 & L' \end{Bmatrix} P_{l_2 l_4}^{l_1 l_3}(L') + (-1)^{L+L'} \begin{Bmatrix} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{Bmatrix} P_{l_3 l_2}^{l_1 l_4}(L') \right], \quad (37)$$

where $\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix}$ is the Wigner 6j-symbol, and

$$P_{l_3 l_4}^{l_1 l_2}(L) = \mathcal{T}_{l_3 l_4}^{l_1 l_2}(L) + (-1)^{l_1+l_2+L} \mathcal{T}_{l_3 l_4}^{l_2 l_1}(L) + (-1)^{l_3+l_4+L} \mathcal{T}_{l_4 l_3}^{l_1 l_2}(L) + (-1)^{l_1+l_2+l_3+l_4} \mathcal{T}_{l_4 l_3}^{l_2 l_1}(L). \quad (38)$$

The following properties of spherical harmonics [45] are useful for our purpose:

$$\nabla^2 Y_l^m(\Omega) = -l(l+1) Y_l^m(\Omega), \quad (39)$$

$$\int d\Omega Y_{l_1}^{m_1}(\Omega) Y_{l_2}^{m_2}(\Omega) = (-1)^{m_1} \delta_{l_1 l_2} \delta_{m_1, -m_2}, \quad (40)$$

$$Y_{l_1}^{m_1}(\Omega) Y_{l_2}^{m_2}(\Omega) = \sum_{l_3, m_3} (-1)^{m_3} I_{l_1 l_2 l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y_{l_3}^{m_3}(\Omega), \quad (41)$$

where

$$I_{l_1 l_2 l_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (42)$$

This quantity $I_{l_1 l_2 l_3}$ is completely symmetric with respect to permutations of its arguments, non-zero only when l_1, l_2, l_3 satisfy the triangular inequality and $l_1 + l_2 + l_3$ is an even number [45]. With repeated use of Eqs. (39)–(41), one can calculate the required parameters. Introducing a notation,

$$\{l\} \equiv l(l+1), \quad (43)$$

the results are

$$\sigma_j^2 = \frac{1}{4\pi} \sum_l (2l+1) \{l\}^j C_l W_l^2, \quad (44)$$

$$S_A = \frac{1}{4\pi\sigma^4} \sum_{l_1, l_2, l_3} I_{l_1 l_2 l_3} \tilde{S}_A^{l_1 l_2 l_3} B_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}, \quad (45)$$

$$K_A = \frac{1}{4\pi\sigma^6} \sum_{l_1, l_2, l_3, l_4, L} \frac{I_{l_1 l_2 l_3 l_4 L}}{2L+1} \tilde{K}_A^{l_1 l_2 l_3 l_4}(L) T_{l_3 l_4}^{l_1 l_2}(L) W_{l_1} W_{l_2} W_{l_3} W_{l_4}, \quad (46)$$

where

$$\tilde{S}_{l_1 l_2 l_3} = 1, \quad \tilde{S}_{l_1 l_2 l_3} = -\frac{\{l_1\} + \{l_2\} + \{l_3\}}{6q^2}, \quad (47)$$

$$\tilde{S}_{\text{II} l_1 l_2 l_3} = \frac{\{l_1\}^2 + \{l_2\}^2 + \{l_3\}^2 - 2\{l_1\}\{l_2\} - 2\{l_2\}\{l_3\} - 2\{l_3\}\{l_1\}}{12q^4}, \quad (48)$$

$$\tilde{K}_{l_3 l_4}^{l_1 l_2}(L) = 1, \quad \tilde{K}_{l_3 l_4}^{l_1 l_2}(L) = -\frac{\{l_1\} + \{l_2\} + \{l_3\} + \{l_4\}}{8q^2}, \quad (49)$$

$$\tilde{K}_{\text{II} l_3 l_4}^{l_1 l_2}(L) = \frac{\{L\}^2 - (\{l_1\} + \{l_2\})(\{l_3\} + \{l_4\})}{16q^4}, \quad (50)$$

$$\tilde{K}_{\text{III} l_3 l_4}^{l_1 l_2}(L) = \frac{(\{l_1\} + \{l_2\} - \{L\})(\{l_3\} + \{l_4\} - \{L\})}{32q^4}. \quad (51)$$

The above forms of skewness parameters [Eq. (47), (48)] were already appeared in [35].

Resemblances of the above results to those of the flat space are obvious if the integrands of Eqs. (30)–(32) are symmetrized with respect to l_1, l_2, l_3 , and l_4 [conversely, one can desymmetrize the Eqs. (47)–(51) to have the similar form with Eqs. (30)–(32)]. Noting the all-sky and flat-sky correspondence [44, 46, 47], it is a straightforward exercise to show that the above all-sky equations reduce to those of flat-sky in the large- l limit. Following [44, 46, 47], but applying an improved approximation

$$Y_l^m(\theta, \phi) \approx (-1)^m \sqrt{\frac{2l+1}{4\pi}} J_m \left[\left(l + \frac{1}{2} \right) \theta \right] e^{im\phi}, \quad (52)$$

for $\theta \ll 1$, $l \gg 1$, the correspondences between all-sky and flat-sky spectra are derived as

$$C_l \approx C \left(l + \frac{1}{2} \right), \quad (53)$$

$$B_{l_1 l_2 l_3} \approx I_{l_1 l_2 l_3} B \left(l_1 + \frac{1}{2}, l_2 + \frac{1}{2}, l_3 + \frac{1}{2} \right), \quad (54)$$

$$P_{l_3 l_4}^{l_1 l_2}(L) \approx I_{l_1 l_2 l_3 l_4 L} \times P \left(l_1 + \frac{1}{2}, l_2 + \frac{1}{2}; l_3 + \frac{1}{2}, l_4 + \frac{1}{2}; L + \frac{1}{2} \right), \quad (55)$$

and

$$T(l_1, l_2, l_3, l_4; l_{12}, l_{23}) = P(l_1, l_2; l_3, l_4; l_{12}) + P(l_1, l_3; l_2, l_4; l_{13}) + P(l_1, l_4; l_2, l_3; l_{23}), \quad (56)$$

where $l_{13} = [l_1^2 + l_2^2 + l_3^2 + l_4^2 - l_{12}^2 - l_{23}^2]^{1/2}$.

Since the all-sky multipole ℓ and the flat-sky wavenumber $|l|$ are related by $|l| = \ell + 1/2$, contributions of the multipole ℓ to the all-sky summation is approximately represented by the flat-sky integration over the range $\ell - 1/2 \leq |l| - 1/2 < \ell + 1/2$, i.e., $\ell \leq |l| < \ell + 1$. Thus, all-sky summations over $\ell = 2, 3, \dots$ correspond to the flat-sky integrations with the limit $|l| \geq 2$, as noted above. We confirm that the flat-sky approximations of Eqs. (24)–(26) with the above correspondences numerically reproduce the values calculated from all-sky formula of Eqs. (44)–(46) within several percent for $\theta_s < 100^\circ$.

For numerical evaluations of the kurtosis and its derivatives by the summation of Eq. (46), the number of terms to add is of order $O(l_{\text{max}}^6)$, where l_{max} is the maximum multipole required for a given smoothing scale, e.g., $l_{\text{max}} \sim \text{several} \times \theta_s^{-1}$. The computational cost becomes progressively high for large l_{max} , if the summation is naively performed. Efficient evaluations are necessary when the smoothing angle θ_s is small. In Appendix A, numerical schemes for the efficient evaluations are summarized.

IV. A SIMPLE EXAMPLE: THE LOCAL MODEL OF NON-GAUSSIANITY

The analytic MFs are evaluated once the power spectrum, bispectrum and trispectrum are given. These spectra depend on models of primordial density fluctuations. As a simple example, we consider below the local model of non-Gaussianity, although our formulas are not restricted to this particular model.

In the local model, the primordial curvature perturbations during the matter era is assumed to take the form [42, 43, 48–50]

$$\Phi(\mathbf{x}) = \phi(\mathbf{x}) + f_{\text{NL}} (\phi^2(\mathbf{x}) - \langle \phi^2 \rangle) + g_{\text{NL}} \phi^3(\mathbf{x}), \quad (57)$$

in configuration space, where ϕ is an auxiliary random Gaussian field. The comoving curvature perturbation ζ is given by $\zeta = 3\Phi/5$. The CMB fluctuations generated by the curvature perturbations have the harmonic coefficients

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \tilde{\Phi}(\mathbf{k}) g_l^{(\text{T})}(\mathbf{k}) Y_l^{m*}(\hat{\mathbf{k}}), \quad (58)$$

where $\tilde{\Phi}(\mathbf{k})$ is the Fourier transform of the primordial curvature perturbation $\Phi(\mathbf{x})$, and $g_l^{(\text{T})}(\mathbf{k})$ is the radiation transfer function.

The bispectrum and trispectrum of CMB in the local model of Eq. (57) are derived in literature [43, 49]:

$$B_{l_1 l_2 l_3} = 2f_{\text{NL}} I_{l_1 l_2 l_3} \left[\int r^2 dr \alpha_{l_1}(r) \beta_{l_2}(r) \beta_{l_3}(r) + \text{cyc.} \right], \quad (59)$$

$$\begin{aligned} \mathcal{T}_{l_3 l_4}^{l_1 l_2}(L) = & I_{l_1 l_2 l_3 l_4 L} \\ & \times \left\{ 4f_{\text{NL}}^2 \int r_1^2 dr_1 r_2^2 dr_2 F_L(r_1, r_2) \alpha_{l_1}(r_1) \beta_{l_2}(r_1) \alpha_{l_3}(r_2) \beta_{l_4}(r_2) \right. \\ & \left. + g_{\text{NL}} \int r^2 dr \beta_{l_2}(r) \beta_{l_4}(r) [\alpha_{l_1}(r) \beta_{l_3}(r) + \beta_{l_1}(r) \alpha_{l_3}(r)] \right\} \quad (60) \end{aligned}$$

where

$$F_L(r_1, r_2) \equiv 4\pi \int \frac{k^2 dk}{2\pi^2} P_\phi(k) j_L(kr_1) j_L(kr_2), \quad (61)$$

$$\alpha_l(r) \equiv 4\pi \int \frac{k^2 dk}{2\pi^2} g_l^{(\text{T})}(k) j_l(kr), \quad (62)$$

$$\beta_l(r) \equiv 4\pi \int \frac{k^2 dk}{2\pi^2} P_\phi(k) g_l^{(\text{T})}(k) j_l(kr), \quad (63)$$

and $P_\phi(k) \propto k^{n_s-4}$ is the primordial power spectrum of ϕ . In some extended models, the factor $4f_{\text{NL}}^2$ in the trispectrum is replaced by $4f_{\text{NL}}^2 \rightarrow 25\tau_{\text{NL}}/9$, and τ_{NL} is considered as an independent parameter [51].

The variance parameters σ and σ_1 are also affected by non-Gaussianity. Therefore we need to evaluate the corrections to the power spectrum. The non-Gaussian corrections to the parameters σ , σ_1 mostly affect the normalization of MFs. In practice, the normalization suffers contamination from observational effects such as pixelization and/or boundary effects, and therefore is not usually used for extracting cosmological information. Nevertheless, we include the effect for theoretical consistency here. By substituting Eq. (58) into Eq. (34) for a local model of Eq. (57), performing angular integrations, we obtain

$$C_l = \left[1 + 6g_{\text{NL}} \int r^2 dr \xi_\phi(r) \right] \tilde{C}_l + 2f_{\text{NL}}^2 \int r^2 dr \gamma_l(r) [\xi_\phi(r)]^2, \quad (64)$$

where

$$\tilde{C}_l = 4\pi \int \frac{k^2 dk}{2\pi^2} P_\phi(k) [g_l^{(\text{T})}(k)]^2, \quad (65)$$

$$\xi_\phi(r) = \int \frac{k^2 dk}{2\pi^2} P_\phi(k) j_0(kr), \quad (66)$$

$$\gamma_l(r) = (4\pi)^2 \int \frac{k^2 dk}{2\pi^2} [g_l^{(\text{T})}(k)]^2 j_0(kr). \quad (67)$$

The quantities \tilde{C}_l , $\xi_\phi(r)$ are the angular power spectrum from the Gaussian component, and the spatial correlation function of ϕ , respectively. Substituting Eqs. (59), (60), (64) into Eqs. (44)–(46) with Eqs. (37), (38), all the parameters necessary to calculate the analytic MFs are evaluated for a model of local non-Gaussianity with a full transfer function $g_l^{(\text{T})}(k)$.

For the Sachs-Wolfe effect [52], which is valid only for small multipole moments ($l \ll 100$), the radiation transfer function is simply given by $g_l^{(\text{T})}(k) = j_l(kr_*)/3$, where r_* is the comoving distance to the last scattering surface. In this limit, the reduced bispectrum and trispectrum have the forms [6, 43, 49]

$$B_{l_1 l_2 l_3} = -6f_{\text{NL}} (C_{l_1}^{\text{SW}} C_{l_2}^{\text{SW}} + C_{l_2}^{\text{SW}} C_{l_3}^{\text{SW}} + C_{l_3}^{\text{SW}} C_{l_1}^{\text{SW}}) I_{l_1 l_2 l_3}, \quad (68)$$

$$\mathcal{T}_{l_3 l_4}^{l_1 l_2}(L) = 9C_{l_2}^{\text{SW}} C_{l_4}^{\text{SW}} \times [4f_{\text{NL}}^2 C_L^{\text{SW}} + g_{\text{NL}} (C_{l_1}^{\text{SW}} + C_{l_3}^{\text{SW}})] I_{l_1 l_2 L} I_{l_3 l_4 L}, \quad (69)$$

where

$$C_l^{\text{SW}} = \frac{2}{9\pi} \int k^2 dk P_\phi(k) [j_l(kr_*)]^2, \quad (70)$$

is the power spectrum of the Sachs-Wolfe effect from the Gaussian component. In the same limit, non-Gaussian corrections to the power spectrum are also obtained after some

calculation:

$$C_l = \left[1 + \frac{27g_{\text{NL}}}{2\pi} \sum_L (2L+1) C_L^{\text{SW}} \right] C_l^{\text{SW}} + \frac{18f_{\text{NL}}^2}{2l+1} \sum_{l_1, l_2} I_{l_1 l_2 l}^2 C_{l_1}^{\text{SW}} C_{l_2}^{\text{SW}}. \quad (71)$$

V. COMPARISONS WITH NUMERICAL MINKOWSKI FUNCTIONALS

To illustrate how well our formulas work, we compare the analytic MFs with numerical MFs from realizations of the non-Gaussian CMB map. Since the purpose of this comparison is to show the performance of analytic formulas, we simply consider the non-Gaussian map in the Sachs-Wolfe limit. Implementations with a full radiation transfer function will be used in future applications to real data. In the Sachs-Wolfe limit, the temperature fluctuations are given by $\Delta T_{\text{SW}}(\theta, \phi)/T = -\Phi(r_*, \theta, \phi)/3$, and thus generated by the nonlinear mapping of Gaussian fluctuations $\Delta T_{\text{G}}/T$ at each position in the sky:

$$\frac{\Delta T_{\text{SW}}}{T} = \frac{\Delta T_{\text{G}}}{T} - 3f_{\text{NL}} \left(\frac{\Delta T_{\text{G}}}{T} \right)^2 + 9g_{\text{NL}} \left(\frac{\Delta T_{\text{G}}}{T} \right)^3 - (\text{monopole} + \text{dipole}), \quad (72)$$

where monopole and dipole components are subtracted from the resulting map. We use the HEALPix package [53] to generate the non-Gaussian map, apply the smoothing window function, and calculate the first- and second-derivatives of the temperature field. The MFs are evaluated from the field derivatives using the numerical algorithm given in [27, 35]. We assume a simple Sachs-Wolfe power spectrum of $l(l+1)C_l^{\text{SW}}/2\pi = 10^{-10}$ for $l \leq 128$ and $C_l^{\text{SW}} = 0$, otherwise. Without this cut-off, the resulting power spectrum would logarithmically diverge in Eq. (71) (the actual power spectrum has natural damping in high- l regime). We adopt the Gaussian filter $W_l = e^{-l(l+1)\theta_s^2/2}$ with the smoothing radius of $\theta_s = 100'$, and non-Gaussianity parameters $f_{\text{NL}} = 10^2$ and $g_{\text{NL}} = 10^6$. We generate 100,000 realizations of the non-Gaussian map, numerically calculate MFs in each realization, and finally average over the realizations.

In Fig. 1, the analytic and numerical MFs are compared. MFs as functions of the threshold ν are plotted in upper panels. As noted in [35], precise values of overall normalizations are affected by numerical artifacts such as pixelization, boundary effects etc. even in the Gaussian random field at the sub-percent level. However, deviations from the Gaussian shape of MFs as functions of the threshold ν are in very good agreement between numerical and analytic MFs. In lower panels, differences between non-Gaussian MFs and Gaussian MFs, divided by the maximum amplitudes, are compared. For numerical data, MFs from Gaussian realizations are calculated and subtracted from those of non-Gaussian realizations. We find the agreements are extremely well. Note that there is no fitting parameter at all in the comparisons.

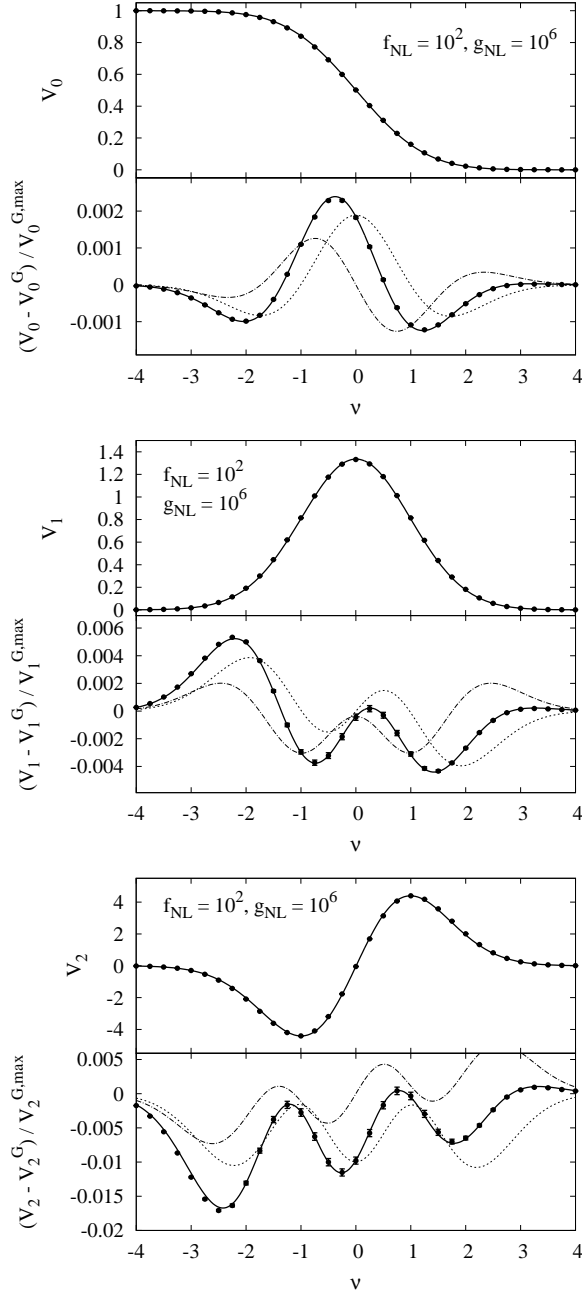


FIG. 1: Comparisons between analytic MFs (solid lines) and numerical MFs (symbols) in the Sachs-Wolfe limit. We use $\theta_s = 100'$ and $f_{\text{NL}} = 10^2$, $g_{\text{NL}} = 10^6$. Contributions from f_{NL} (dotted lines) and g_{NL} (dot-dashed lines) are also shown. Total MFs are given by sums of the two contributions in such a weak non-Gaussianity regime.

Contributions from the parameters f_{NL} and g_{NL} are shown in thin dotted lines and dot-dashed lines, respectively. Phases of oscillating patterns are different from one another. This is because the higher-order non-Gaussianity involves higher-order Hermite polynomials as given in Eqs. (7) and (17)–(19). Since the Hermite polynomials are orthogonal functions, the effect of each parameter can be distinguished with such observations.

VI. SUMMARY

In this paper, the analytic MFs with second-order non-Gaussianity, including effects from the bispectrum and trispectrum, have been derived. As long as the higher-order cumulants obey hierarchical structure, $\langle f^n \rangle_c \sim O(\sigma^{2n-2})$, deviations from the Gaussian predictions of MFs are expanded by σ . First several cumulants are relevant to the non-Gaussian corrections. Up to the second-order in σ , the non-Gaussian corrections are expressed by the skewness, kurtosis, and their derivatives, and thus by the bispectrum and trispectrum. These fundamental results are general and applicable to any 2D random field which has hierarchical orderings of higher-order cumulants. In cosmology, applications to the CMB temperature fluctuations are of great importance. Once the form of bispectrum and trispectrum in CMB are given by a model of the early universe, the analytic MFs are evaluated by our formulas. We have illustrated how the analytic MFs of CMB are calculated, employing the local model of non-Gaussianity, as an example. We have compared the analytical MFs with numerical calculations of non-Gaussian MFs, and found very good agreements. Applications to real data of CMB map are now in progress and will be reported in future work.

Acknowledgments

I wish to thank C. Hikage for discussion and comments. I acknowledge support from the Ministry of Education, Culture, Sports, Science, and Technology, Grant-in-Aid for Scientific Research (C), 21540263, 2009, and Grant-in-Aid for Scientific Research on Priority Areas No. 467 “Probing the Dark Energy through an Extremely Wide and Deep Survey with Subaru Telescope.” This work is supported in part by JSPS (Japan Society for Promotion of Science) Core-to-Core Program “International Research Network for Dark Energy.”

Appendix A: Evaluations of Skewness, Kurtosis and Their Derivatives

When the Eq. (46) is straightforwardly evaluated, the number of terms to add is of order $O(l_{\text{max}}^6)$, where l_{max} is the maximum multipole moment required for a given smoothing scale, e.g., $l_{\text{max}} \sim \text{several} \times \theta_s^{-1}$. For parameters K and K_I , summations over L for terms with $6j$ -symbols can be analytically performed, but we still need numerical summations of $6j$ -symbols for parameters K_{II} and K_{III} . The computational cost is progressively high for large l_{max} , and therefore efficient calculations are necessary when the smoothing angle is small.

First of all, one should take advantages of symmetries and constraints to save the number of additions. For the skewness parameters, the number of addition is reduced by a factor of about six, using the substitution

$$\sum_{l_1, l_2, l_3} = \sum_{l_1=l_2=l_3} + 3 \sum_{l_1=l_2 < l_3} + 3 \sum_{l_1 < l_2=l_3} + 6 \sum_{l_1 < l_2 < l_3}, \quad (\text{A1})$$

in Eq. (45). The summation is taken only when $l_1 + l_2 + l_3$ is an even number and l_1, l_2, l_3 satisfy the triangular inequality, $l_2 - l_1 \leq l_3 \leq l_1 + l_2$ because of the factor $I_{l_1 l_2 l_3}$. For the kurtosis parameters, the number of addition is reduced by a factor of about eight, using the substitution

$$\begin{aligned} \sum_{l_1, l_2, l_3, l_4} = & \sum_{\substack{l_1 = l_2 \\ l_3 = l_4 \\ l_1 = l_3}} + 2 \sum_{\substack{l_1 = l_2 \\ l_3 = l_4 \\ l_1 < l_3}} + 2 \sum_{\substack{l_1 = l_2 \\ l_3 < l_4 \\ l_1 = l_3}} + 2 \sum_{\substack{l_1 < l_2 \\ l_3 = l_4 \\ l_1 = l_3}} \\ & + 4 \sum_{\substack{l_1 = l_2 \\ l_3 < l_4 \\ l_1 < l_3}} + 4 \sum_{\substack{l_1 < l_2 \\ l_3 = l_4 \\ l_1 < l_3}} + 4 \sum_{\substack{l_1 < l_2 \\ l_3 < l_4 \\ l_1 = l_3}} + 8 \sum_{\substack{l_1 < l_2 \\ l_3 < l_4 \\ l_1 < l_3}}, \quad (\text{A2}) \end{aligned}$$

in Eq. (46). The summation is taken only when $l_1 + l_2 + l_3 + l_4$ is an even number and $l_4 \leq l_1 + l_2 + l_3$ is satisfied because of the factor $I_{l_1 l_2 l_3 l_4}$. The index L in Eq. (46) runs over every other integers in the range $[\max(|l_1 - l_2|, |l_3 - l_4|), \min(l_1 + l_2, l_3 + l_4)]$ with the same parity as that of $l_1 + l_2$.

Secondly, since the trispectrum is not a strongly oscillating function in usual cases, one can sparsely sample the multipoles in the sum for large values of l_1, \dots, l_4 . Acoustic oscillations in the CMB are sufficiently mild in this respect. There is a caveat that the summation over L' in Eq. (37) should not be sparsely sampled, since the $6j$ -symbol is a strongly oscillating function with respect to its arguments. Sparse samplings for

large l_i 's with appropriate weights save enormous amounts of time for the calculation.

Instead of calculating the exact summation, one can use the flat-sky approximation of Eq. (26), and evaluate the five-dimensional integral by, e.g., the Monte-Carlo algorithm. Once the reduced trispectrum $\mathcal{T}_{l_3 l_4}^{l_1 l_2}(L)$ is given, Eqs. (38), (55), (56) give the trispectrum $T(l_1, l_2, l_3, l_4; l_{12}, l_{23})$ in the flat-sky approximation. There is no need for summing $6j$ -symbols in this procedure. The integration ranges of Eq. (26) are $2 \leq l_1, l_2, l_3, l_4 < \infty$, $0 \leq \theta_{12}, \theta_{23} \leq 2\pi$ when the monopole and dipole components are subtracted from the map. This approximation is valid when most contributions to the kurtosis parameters come from the flat-sky regime ($l \gg 1$). In realistic spectra, this condition is satisfied because multipoles near the acoustic peak at $l \sim 200$ dominantly contribute. Strictly speaking, the flat-sky integration is valid when all l 's are large. Therefore, for a more precise approximation, one can use the all-sky summation of Eq. (46) only when, e.g., $l_1 \leq 20$ [note that $l_1 \leq l_2, l_3, l_4$ in Eq. (A2)], and otherwise adopt the flat-sky integration of Eq. (26) over the range $l_1, l_2, l_3, l_4 \geq 21$.

Unless sub-percent level accuracies are required, experiences show that the simplest flat-sky integrals over all ranges of multipole with appropriate choice of integration limits suffice for evaluations of skewness and kurtosis parameters in realistic spectra.

-
- [1] V. F. Mukhanov and G. V. Chibisov, Soviet Journal of Experimental and Theoretical Physics Letters **33**, 532 (1981).
 - [2] A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982).
 - [3] S. W. Hawking, Phys. Lett. B **115**, 295 (1982).
 - [4] A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
 - [5] T. J. Allen, B. Grinstein and M. B. Wise, Phys. Lett. B **197**, 66 (1987).
 - [6] A. Gangui, F. Lucchin, S. Matarrese, and S. Mollerach, Astrophys. J. **430**, 447 (1994).
 - [7] D. H. Lyth and D. Wands, Phys. Lett. B **524**, 5 (2002).
 - [8] A. Linde and V. Mukhanov, Phys. Rev. D **56**, R535 (1997).
 - [9] D. H. Lyth, C. Ungarelli and D. Wands, Phys. Rev. D **67**, 023503 (2003).
 - [10] C. Armendáriz-Picón, T. Damour and V. Mukhanov, Phys. Lett. B **458**, 209 (1999).
 - [11] J. Garriga and V. F. Mukhanov, Phys. Lett. B **458**, 219 (1999).
 - [12] M. Alishahiha, E. Silverstein and D. Tong, Phys. Rev. D **70**, 123505 (2004).
 - [13] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D **64**, 123522 (2001).
 - [14] N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, Phys. Rep. **402**, 103 (2004).
 - [15] E. Komatsu, et al., Astrophys. J. Suppl. Ser. **148**, 119 (2003).
 - [16] D. N. Spergel, et al., Astrophys. J. Suppl. Ser. **170**, 377 (2007).
 - [17] P. Creminelli, L. Senatore, M. Zaldarriaga and M. Tegmark, J. Cosmol. Astropart. Phys. **3**, 5 (2007).
 - [18] E. Komatsu, et al., Astrophys. J. Suppl. Ser. **180**, 330 (2009).
 - [19] K. M. Smith, L. Senatore and M. Zaldarriaga, J. Cosmol. Astropart. Phys. **9**, 6 (2009).
 - [20] A. Curto, E. Martínez-González, P. Mukherjee, R. B. Barreiro, F. K. Hansen, M. Liguori and S. Matarrese, Mon. Not. R. Astron. Soc. **393**, 615 (2009).
 - [21] M. Sasaki, J. Väliviita and D. Wands, Phys. Rev. D **74**, 103003 (2006).
 - [22] Q.-G. Huang, J. Cosmol. Astropart. Phys. **6**, 35 (2009).
 - [23] C. T. Byrnes and G. Tasinato, J. Cosmol. Astropart. Phys. **8**, 16 (2009).
 - [24] J.-L. Lehnert, S. Renaux-Petel, Phys. Rev. D **80**, 063503 (2009).
 - [25] K. R. Mecke, T. Buchert and H. Wagner, Astron. Astrophys. **288**, 697 (1994).
 - [26] J. Schmalzing, and T. Buchert, Astrophys. J. Letters **482**, L1 (1997).
 - [27] J. Schmalzing, and K. M. Gorski, Mon. Not. R. Astron. Soc. **297**, 355 (1998).
 - [28] D. Novikov, J. Schmalzing V. F. Mukhanov, Astron. Astrophys. **364**, 17 (2000).
 - [29] J. H. P. Wu, A. Balbi, J. Borrill, P. G. Ferreira, S. Hanany, A. H. Jaffe, A. T. Lee, B. Rabbii, P. L. Richards, G. F. Smoot, R. Stompom and C. D. Winant, Phys. Rev. Lett. **87**, 251303 (2001).
 - [30] G. Polenta, et al., Astrophys. J. Letters **572**, L27 (2002).
 - [31] A. Curto, J. Aumont, J. F. Macías-Pérez, E. Martínez-González, R. B. Barreiro, D. Santos, F. X. Désert and M. Tristram, Astron. Astrophys. **474**, 23 (2007).
 - [32] G. de Troia, et al., Astrophys. J. Letters **670**, L73 (2007).
 - [33] A. Curto, J. F. Macías-Pérez, E. Martínez-González, R. B. Barreiro, D. Santos, F. K. Hansen, M. Liguori and S. Matarrese, Astron. Astrophys. **486**, 383 (2008).
 - [34] T. Matsubara, Astrophys. J. **584**, 1 (2003)

- [35] C. Hikage, E. Komatsu, and T. Matsubara, *Astrophys. J.* **653**, 11 (2006).
- [36] C. Hikage, T. Matsubara, P. Coles, M. Liguori, F. K. Hansen and S. Matarrese, *Mon. Not. R. Astron. Soc.* **389**, 1439 (2008).
- [37] P. Natoli, et al., arXiv:0905.4301 (2009).
- [38] C. Hikage, K. Koyama, T. Matsubara, T. Takahashi, M. Yamaguchi, *Mon. Not. R. Astron. Soc.* **398**, 2188 (2009).
- [39] P. Chingangbam, and, C. Park, *Journal of Cosmology and Astro-Particle Physics* **12**, 19 (2009).
- [40] D. Pogosyan, C. Gay, and C. Pichon, *Phys. Rev. D* **80**, 081301(R) (2009).
- [41] H. Tomita, *Progr. Theor. Phys.* **76**, 952 (1986).
- [42] L. Verde, L. Wang, A. F. Heavens, and M. Kamionkowski, *Mon. Not. R. Astron. Soc.* **313**, 141 (2000).
- [43] E. Komatsu, and D. N. Spergel, *Phys. Rev. D* **63**, 063002 (2001).
- [44] W. Hu, *Phys. Rev. D* **64**, 083005 (2001).
- [45] A. Messiah, *Quantum Mechanics*, Vol.2 (Amsterdam: North-Holland, 1976)
- [46] W. Hu, *Phys. Rev. D* **62**, 043007 (2000).
- [47] M. White, J. E. Carlstrom, M. Dragovan, and W. L. Holzapfel, *Astrophys. J.* **514**, 12 (1999).
- [48] D. S. Salopek, and J. R. Bond, *Phys. Rev. D* **42**, 3936 (1990).
- [49] T. Okamoto, and W. Hu, *Phys. Rev. D* **66**, 063008 (2002).
- [50] N. Kogo, and E. Komatsu, *Phys. Rev. D* **73**, 083007 (2006).
- [51] C. T. Byrnes, M. Sasaki and D. Wands, *Phys. Rev. D* **74**, 123519 (2006).
- [52] R. K. Sachs, and A. M. Wolfe, *Astrophys. J.* **147**, 73 (1967).
- [53] K. M. Górski, E. Hivon, A. J. Banday, B. D. Wandelt, F. K. Hansen, M. Reinecke, and M. Bartelmann, *Astrophys. J.* **622**, 759 (2005).